Operator Algebras Generated by Left Invertibles

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Background

- A sequence $\{f_n\}$ in a Hilbert space $\mathscr{H}$ is called a **frame** if there exists constants $0 < A < B$ such that for each $x \in \mathscr{H}$,

$$A\|x\|^2 \leq \sum_n |\langle x, f_n\rangle|^2 \leq B\|x\|^2$$

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- We can associate to each frame $\{f_n\}$ a dual frame $\{g_n\}$ such that

$$x = \sum_n \langle x, g_n \rangle f_n$$

Remark

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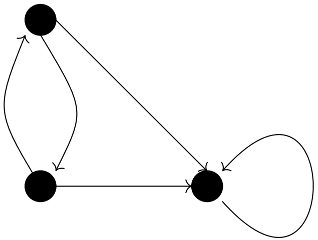
$$E = \{r, s, E^0, E^1\}:$$



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Definition

Let $T \in \mathscr{B}(\mathscr{H})$ have closed range. There is a unique operator $T^\dagger \in \mathscr{B}(\mathscr{H})$ called the **Moore-Penrose inverse of T** such that

1. $T^\dagger T x = x$ for all $x \in \ker(T)^\perp$
2. $T^\dagger y = 0$ for all $y \in (T\mathscr{H})^\perp$.

Example

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- Let $T \in \mathscr{B}(\ell^2)$ be given by $Te_n = w_n e_n$, $n \geq 0$. If $0 < c < |w_n|$, then T is left invertible and

$$T^\dagger e_n = \begin{cases} 0 & n = 0 \\ w_n^{-1} e_{n-1} & n \geq 1 \end{cases}$$

Program

For each edge e in Γ, pick operators $\{T_e\}_{e\in E^1}$ with closed range subject to constraints of graph. Analyze the structure of the operator algebra

$$\mathfrak{A}_\Gamma := \overline{\mathrm{Alg}}(\{T_e, T_e^\dagger\}_{e\in E^1}).$$

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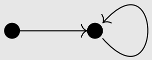
Remark

Our focus is on representations afforded by the graph



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Let T be a left invertible operator, and $T^\dagger$ its Moore-Penrose inverse. Set

$$\mathfrak{A}_T := \overline{\mathrm{Alg}}(T, T^\dagger).$$

Question

1. In what way does $\mathfrak{A}_T$ look like the C*-algebra generated by an isometry?
2. What are the isomorphism classes of $\mathfrak{A}_T$?

Example

If $T = M_z$ on $H^2(\mathbb{T})$, then $\mathfrak{A}_T$ is the classic Toeplitz algebra

$$\mathcal{T} = \{T_f + K : f \in C(\mathbb{T}), K \in \mathscr{K}(H^2(\mathbb{T}))\}$$

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Remark

General left invertibles have no Wold decomposition:

$$\mathscr{H} \neq \left(\bigcap_n T^n \mathscr{H}\right) \oplus \left(\bigvee_n T^n \ker(T^*)\right)$$

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Theorem (D-)

Let T be an analytic left invertible with ind$(T) = -n$ for some positive integer n. Let $\{x_{i,0}\}_{i=1}^n$ be an orthonormal basis for $\ker(T^*)$. *Then*

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$$x_{i,j} := (T^{\dagger^*})^j (x_{i,0})$$

$i = 1, \dots n$, $j = 0, 1, \dots$ *is a Schauder basis for* $\mathscr{H}$

Definition

An operator $R \in \mathscr{B}(\mathscr{H})$ is called **Cowen-Douglas** if there exists open subset $\Omega \subset \sigma(R)$ such that

1. $(R-\lambda)\mathscr{H} = \mathscr{H}$ for all $\lambda \in \Omega$
2. $\dim(\ker(R-\lambda)) = n$ for all $\lambda \in \Omega$.
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3. *There exists $\epsilon > 0$ such that $T^\dagger \in B_n(\Omega)$ for $\Omega = \{z : |z| < \epsilon\}$*

Theorem (Zhu)

If $R \in B_n(\Omega)$, then R is unitarily equivalent to M_z^* on a RKHS of analytic functions $\widehat{\mathscr{H}}$ on $\Omega^* = \{\overline{z} : z \in \Omega\}$.

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Analytic Model

Let T be an analytic left invertible with $\operatorname{ind}(T) = -n$ for some positive integer n, $\{x_{i,j}\}$ the basis associated with ${T^{\dagger}}^*$, and $\Omega = \{z : |z| < \epsilon\}$ as in previous theorem.

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$$\gamma(\lambda) := \sum_{i=1}^{n} \phi_i(\lambda) \sum_{j \geq 0} \lambda^j x_{i,j}$$

exists in $\mathscr{H}$ for each $\lambda \in \Omega$.

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$$\gamma(\lambda) := \sum_{i=1}^{n} \phi_i(\lambda) \sum_{j \geq 0} \lambda^j x_{i,j}$$

exists in $\mathscr{H}$ for each $\lambda \in \Omega$. Moreover, for each $f \in \mathscr{H}$,

$$\hat{f}(\lambda) = \langle f, \gamma(\overline{\lambda}) \rangle = \sum_{i=1}^{n} \phi_i(\lambda) \sum_{j \geq 0} \lambda^j \langle f, x_{i,j} \rangle.$$

Theorem (D-)

If T is an analytic left invertible with $\mathrm{ind}(T) = -1$, then $\mathfrak{A}_T$ contains the compact operators $\mathscr{K}(\mathscr{H})$. Moreover, $\mathscr{K}(\mathscr{H})$ is a minimal ideal of $\mathfrak{A}_T$.

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Corollary

$I - TT^\dagger, I - T^\dagger T \in \mathscr{K}(\mathscr{H})$. Thus, $\pi(T)^{-1} = \pi(T^\dagger)$. Hence, we have the following:

$$0 \longrightarrow \mathscr{K}(\mathscr{H}) \xrightarrow{\iota} \mathfrak{A}_T \xrightarrow{\pi} \mathscr{B} \longrightarrow 0$$

where $\mathscr{B} = \overline{Alg}\{\pi(T), \pi(T^\dagger)\}$.

Definition

An operator $S \in \mathscr{B}(\mathscr{H})$ is **subnormal** if it has a normal extension:

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Definition

Let μ be a scalar-valued spectral measure associated to N, and $f \in L^\infty(\sigma(N), \mu)$. Define $T_f \in \mathscr{B}(\mathscr{H})$ via

$$T_f := P(f(N)) \,|_{\mathscr{H}}$$

where P is the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$.

Theorem (Keough, Olin and Thomson)

If S is an irreducible, subnormal, essentially normal operator, then:

$$C^*(S) = \{T_f + K : f \in C(\sigma(N)), K \in \mathscr{K}(\mathscr{H})\}$$

Moreover, if $\sigma(N) = \sigma_e(S)$, then each element has $A \in C^*(S)$ has a unique representation of the form $T_f + K$.

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on $\sigma_e(S)$. Then

$$\mathfrak{A}_S = \{T_f + K : f \in \mathscr{B}, K \in \mathscr{K}(\mathscr{H})\}$$

Moreover, the representation of each element as $T_f + K$ is unique.

Theorem (D-)

Let T_i, $i = 1, 2$ be left analytic left invertible with $ind(T_i) = -1$, *and* $\mathfrak{A}_i := \mathfrak{A}_{T_i}$. *Suppose that* $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ *is a bounded isomorphism.*

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Remark

To distinguish these algebras by isomorphism classes, we need to classify the similarity orbit:

$$\mathcal{S}(T) := \{VTV^{-1} : V \in \mathscr{B}(\mathscr{H}) \text{ is invertible}\}$$

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Theorem (Jiang, Wang, Guo, Ji)

Let $A, B \in B_1(\Omega)$. Then A is similar to B if and only if

$$K_0(\{A \oplus B\}') \cong \mathbb{Z}$$

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- No Wold ⇒ analytic: $\bigcap_n T^n \mathscr{H} = 0$
- Analytic ⇒ Cowen-Douglas
  1. Canonical Model
  2. Classification program ⇒ Similarity orbit/K-theoretic obstruction

Future Work:

- Determine the isomorphism classes for $\mathrm{ind}(T) < -1$.

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- Any hope for non-analytic left invertibles?
- Investigate other algebras that arise from graphs - e.g. "Cuntz algebra".