Operator Algebras Generated by Left Invertibles

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Definition

A **Hilbert space** $\mathscr{H}$ is

1. inner product space: $\langle \cdot, \cdot \rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$
2. complete with respect to the norm $\|x\|^2 = \langle x, x \rangle$.

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For $T \in \mathscr{B}(\mathscr{H})$, the **adjoint** $T^* \in \mathscr{B}(\mathscr{H})$ such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for each $x, y \in \mathscr{H}$.

Example

$\mathscr{H} = \mathbb{C}^n$, $\mathscr{B}(\mathbb{C}^n) = M_n$, $(a_{i,j})^* = (\overline{a_{j,i}})$.

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Definition

If $F \in \mathscr{B}(\mathscr{H})$ satisfies $\dim(\operatorname{ran}(F)) < \infty$, F is *finite rank*. An operator $K \in \mathscr{B}(\mathscr{H})$ is called **compact** if K is the norm-limit of finite rank operators. We write

$$\mathscr{K}(\mathscr{H}) := \{\text{all compact operators on } \mathscr{H}\}.$$

Definition

Let

$$\mathscr{H} = \ell^2(\mathbb{N}) = \{(a_1, a_2, \dots) : \sum |a_n|^2 < \infty\}.$$

The **unilateral shift** $S \in \mathscr{B}(\mathscr{H})$ is

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Also,

- $\ker(S) = 0$, $\ker(S^*) = \operatorname{span}\{e_1\}$
- $S^*S = I$
- S is **isometric**: $\|Sx\| = \|x\|$ for all $x \in \mathscr{H}$.

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- V^* models step back

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Remark

C*-algebra's that encode dynamics of groups, groupoids, graphs, etc. are well studied.

Definition

A sequence $\{f_n\}$ in a Hilbert space $\mathscr{H}$ is called a **frame** if there exists constants $0 < A < B$ such that for each $x \in \mathscr{H}$,

$$A\|x\|^2 \leq \sum_n |\langle x, f_n \rangle|^2 \leq B\|x\|^2$$

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We can associate to each frame $\{f_n\}$ a (canonical) dual frame $\{g_n\}$ such that

$$x = \sum_n \langle x, g_n \rangle f_n$$

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Question

What is the analog of the adjoint for a closed range operator?

Definition

Let $T \in \mathscr{B}(\mathscr{H})$ have closed range. There is a unique operator $T^\dagger \in \mathscr{B}(\mathscr{H})$ called the **Moore-Penrose inverse of T** such that

1. $T^\dagger T x = x$ for all $x \in \ker(T)^\perp$
2. $T^\dagger y = 0$ for all $y \in (T\mathscr{H})^\perp$.

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Example

- If T is an isometry, then $T^\dagger = T^*$.
- Let $T \in \mathscr{B}(\ell^2)$ be given by $Te_n = w_n e_{n+1}$, $n \geq 1$. If $0 < c < |w_n|$, then T has closed range (left invertible) and

$$T^\dagger e_n = \begin{cases} 0 & n = 1 \\ w_n^{-1} e_{n-1} & n \geq 2 \end{cases}$$

Program

For each edge e in Γ, pick operators $\{T_e\}_{e\in E^1}$ with closed range subject to constraints of graph. Analyze the structure of the operator algebra

$$\mathfrak{A}_\Gamma := \overline{\mathrm{Alg}}(\{T_e, T_e^\dagger\}_{e\in E^1}).$$

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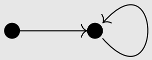
Remark

Our focus is on representations afforded by the graph



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Focus

Let T be a left invertible operator, and $T^\dagger$ its Moore-Penrose inverse. Set

$$\mathfrak{A}_T := \overline{\mathrm{Alg}}(T, T^\dagger).$$

Question

1. In what way does $\mathfrak{A}_T$ look like the C*-algebra generated by an isometry?
2. What are the isomorphism classes of $\mathfrak{A}_T$?

Proposition (Wold-Decomposition)

If $V \in \mathscr{B}(\mathscr{H})$ *is an isometry, then*

$$V = U \oplus (\oplus_{\alpha \in A} S)$$

where U *is a unitary and* S *is the shift operator. Namely,*

$$\mathscr{H} = \left(\bigcap_{n \geq 0} V^n \mathscr{H} \right) \oplus \left(\bigvee_{n \geq 0} V^n \ker(V^*) \right)$$

and $|A| = \dim(\ker(V^*))$.

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Idea

If one wants to analyze $C^*(V)$ for some isometry V, one needs to understand $C^*(S)$.

The functions $e_n(z) := z^n$ for $n \in \mathbb{Z}$ form an orthonormal basis for $L^2(\mathbb{T})$ with normalized Lebesgue measure.

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If $f \in L^\infty(\mathbb{T})$, define $M_f \in \mathscr{B}(L^2(\mathbb{T}))$ via

$$M_f(g) = fg \qquad \forall g \in L^2(\mathbb{T}).$$

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The **Toeplitz operator** $T_f \in \mathscr{B}(H^2(\mathbb{T}))$ is

$$T_f := P_{H^2(\mathbb{T})} M_f \mid_{H^2(\mathbb{T})} .$$

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Theorem (Coburn)

We have

$$C^*(T_z) = \{T_f + K : f \in C(\mathbb{T}), K \in \mathscr{K}(H^2(\mathbb{T}))\}.$$

Moreover, if $A \in C^*(T_z)$, $A = T_f + K$ *for exactly one* $f \in C(\mathbb{T})$ *and* $K \in \mathscr{K}(H^2(\mathbb{T}))$.

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Moreover, if $A \in C^*(T_z)$, $A = T_f + K$ for exactly one $f \in C(\mathbb{T})$ and $K \in \mathscr{K}(H^2(\mathbb{T}))$. Further, $\mathscr{K}(H^2(\mathbb{T}))$ is the unique minimal ideal of $C^*(T_z)$. Also $I - SS^*, I - S^*S \in \mathscr{K}(\mathscr{H})$, yielding

$$0 \longrightarrow \mathscr{K}(H^2(\mathbb{T})) \xrightarrow{\iota} C^*(T_z) \xrightarrow{\pi} C(\mathbb{T}) \longrightarrow 0$$

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General left invertibles have no Wold decomposition:

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$$T = \begin{pmatrix} S & 0 \\ \iota & U \end{pmatrix}$$

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Definition

A left invertible operator T is called **analytic** if

$$\bigcap_n T^n \mathscr{H} = 0.$$

Remark

If V is an analytic isometry ($U = 0$ in Wold-decomposition), $\dim\ker(V^*) = n$ and $\{e_{i,0}\}_{i=1}^n$ is an orthonormal basis for $\ker(V^*)$, then

$$e_{i,j} = V^j(e_{i,0})$$

$i = 1, \ldots n$, $j = 0, 1, \ldots$ is an orthonormal basis for $\mathscr{H}$.

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Let T be an analytic left invertible with $\dim\ker(T^*) = n$ for some positive integer n. Let $\{x_{i,0}\}_{i=1}^{n}$ be an orthonormal basis for $\ker(T^*)$. Then

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$i = 1, \ldots n,\ j = 0, 1, \ldots$ *is a Schauder basis for $\mathscr{H}$.*

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Given $\Omega \subset \mathbb{C}$ open, $n \in \mathbb{N}$, we say that R is **Cowen-Douglas**, and write $R \in B_n(\Omega)$ if

1. $\Omega \subset \sigma(R) = \{\lambda \subset \mathbb{C} : R - \lambda \text{ not invertible}\}$
2. $(R - \lambda)\mathscr{H} = \mathscr{H}$ for all $\lambda \in \Omega$
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3. *There exists $\epsilon > 0$ such that $T^\dagger \in B_n(\Omega)$ for $\Omega = \{z : |z| < \epsilon\}$*

Analytic Model

- If $R \in B_n(\Omega)$, there is a analytic map $\gamma : \Omega \to \mathscr{H}$ such that $\gamma(\lambda) \in \ker(R - \lambda)$.

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- For each $f \in \mathscr{H}$, define a holomorphic function $\hat{f}$ over $\Omega^* := \{\overline{z} : z \in \Omega\}$ via

$$\hat{f}(\lambda) = \langle f, \gamma(\overline{\lambda}) \rangle.$$

Analytic Model

- If $R \in B_n(\Omega)$, there is a analytic map $\gamma : \Omega \to \mathscr{H}$ such that $\gamma(\lambda) \in \ker(R - \lambda)$.
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- Let $\widehat{\mathscr{H}} = \{\hat{f} : f \in \mathscr{H}\}$. Equip with $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$.
- Then $U : \mathscr{H} \to \widehat{\mathscr{H}}$ via $Uf = \hat{f}$ is unitary, and

$$(UTf)(\lambda) = \langle Tf, \gamma(\overline{\lambda}) \rangle = \langle f, \overline{\lambda}\gamma(\overline{\lambda}) \rangle = (M_z U f)(\lambda)$$

Corollary

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If $T \in \mathscr{B}(\mathscr{H})$ is left invertible with $\dim\ker(T^*) = n$, then

$$Alg(T, T^\dagger) = \left\{ F + \sum_{n=0}^{N} \alpha_n T^n + \sum_{m=1}^{M} \beta_m {T^\dagger}^m : F \text{ is finite rank} \right\}.$$

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Heuristic

$\mathfrak{A}_T$ is compact perturbations of of multiplication operators with symbols Laurent series centered at zero.

Theorem (D-)

If T is an analytic left invertible with $\dim\ker(T^*) = 1$, then $\mathfrak{A}_T$ contains the compact operators $\mathscr{K}(\mathscr{H})$. Moreover, $\mathscr{K}(\mathscr{H})$ is a minimal ideal of $\mathfrak{A}_T$.

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Corollary

$I - TT^\dagger, I - T^\dagger T \in \mathscr{K}(\mathscr{H})$. Thus, $\pi(T)^{-1} = \pi(T^\dagger)$. Hence, we have the following:

$$0 \longrightarrow \mathscr{K}(\mathscr{H}) \xrightarrow{\ \iota\ } \mathfrak{A}_T \xrightarrow{\ \pi\ } \mathscr{B} \longrightarrow 0$$

where $\mathscr{B} = \overline{Alg}\{\pi(T), \pi(T^\dagger)\}$.

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  The operator N is said to be a **minimal normal extension** if $\mathscr{K}$ has no proper subspace reducing N and containing $\mathscr{H}$.

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Theorem (Keough, Olin and Thomson)

If S is an irreducible, subnormal, essentially normal operator, such that $\sigma(N) = \sigma_e(S)$. Then

$$C^*(S) = \{T_f + K : f \in C(\sigma_e(S)), K \in \mathscr{K}(\mathscr{H})\}.$$

Moreover, then each element has $A \in C^*(S)$ has a unique representation of the form $T_f + K$.

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Moreover, the representation of each element as $T_f + K$ is unique.

Theorem (D-)

Let T_i, $i = 1, 2$ *be left invertible (analytic,* $\dim\ker(T_i^*) = 1$*) with* $\mathfrak{A}_i := \mathfrak{A}_{T_i}$. *Suppose that* $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ *is a bounded isomorphism.*

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Remark

To distinguish these algebras by isomorphism classes, we need to classify the similarity orbit:

$$\mathcal{S}(T) := \{VTV^{-1} : V \in \mathscr{B}(\mathscr{H}) \text{ is invertible}\}$$

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- Determining the similarity orbit of Cowen-Douglas operators is a classic problem.

Theorem (Jiang, Wang, Guo, Ji)

Let $A, B \in B_1(\Omega)$. Then A is similar to B if and only if

$$K_0(\{A \oplus B\}') \cong \mathbb{Z}$$

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- Investigate other algebras that arise from graphs - e.g. "Cuntz algebra".